Regular Assessment - Part I - 17/12/2024 $\boxed{\text{I}}$

Solutions

1 a) $D_f = \{(x,y) \in \mathbb{R}^2 \quad x \neq \pm 1 \wedge x \neq 0\}$

b) $(2, e^{-2})$ for instance

$\text{int}\ \Omega = \{(x,y) \in \mathbb{R}^2 : \ x \in]1,2[\wedge \ 0 < y < e^{-x}\}$

closed

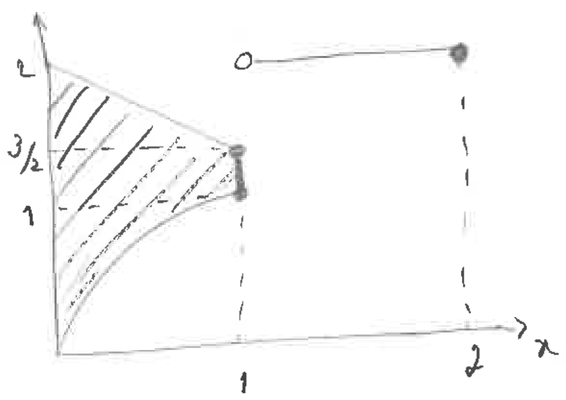
c) $\lim\limits_{n \to +\infty} f(x_n) = -6.$

d) $f(x,y) = x^4 y^2$

e) $f(x) = 6$ for instance

injective

f)



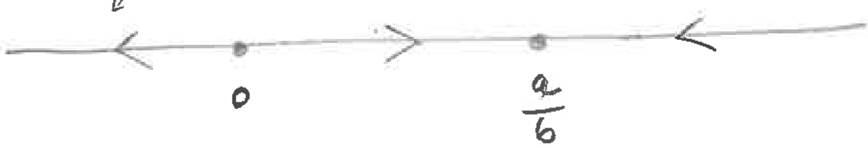
Fix $H = \{0,1,2\}$

g) $H_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ global

h) $-1-2\lambda y = 0$

$\lambda(1-x^2-y^2)=0$

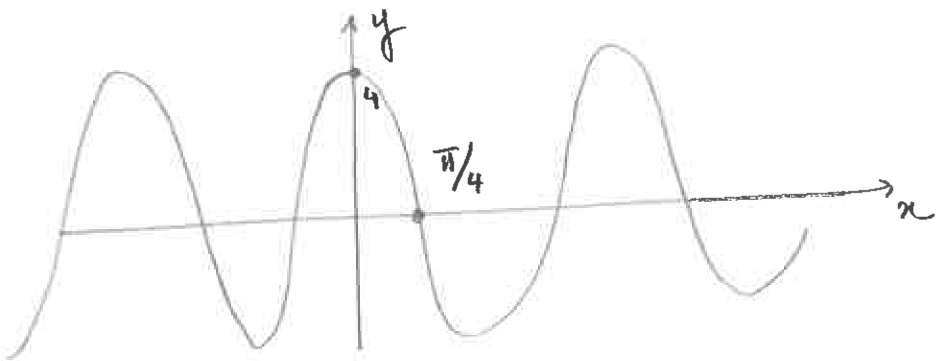
i) (we may omit since $p \geq 0$)



$\lim_{t\to+\infty} y(t) = a/b$

(j) increasing

k) $y(x) = 4\cos(2x)$



e) $\begin{cases} \dot{x} = 2x \\ \dot{y} = 3y \end{cases}$ unstable ; Hartman-Grobman.

m) $\det A < 0$

n) $\min \int_0^{10} \underbrace{\frac{x^2}{2} + \frac{u^2}{2}}_{F} \, dt$ $\qquad g(t,x,u) = u$

$$H(t,x,u,p) = -\frac{x^2}{2} - \frac{u^2}{2} + p \cdot u$$

$$\boxed{\frac{\partial H}{\partial u} = 0 \iff -\frac{2u}{2} + p = 0 \iff p = u}$$

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \iff \begin{cases} \dot{x} = u \\ \dot{p} = -\left(-\frac{\cancel{2}x}{\cancel{2}}\right) \end{cases} \iff \begin{cases} \dot{x} = p \\ \dot{p} = x \end{cases}$$

Regular Assessment - Part II - 17/12/2024

1 a)

- $[1,+\infty[$ is closed $\Rightarrow$ $[1,+\infty[$ is complete

- f is a contraction because:

$$\forall x \in D_f \quad |f'(x)| < 1$$

Proof:

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{4} \qquad |f'(x)| \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$$

1 b) $\lim\limits_{n\to+\infty} f^n(x_0) = p$ where $p = f(p)$ and $x_0 \in [1,+\infty[$.

$$f(x) = x \Leftrightarrow \sqrt{x} + \frac{x}{4} = x \Leftrightarrow \sqrt{x} = \frac{3}{4}x \Leftrightarrow$$

$$\Longrightarrow x = \frac{9}{16}x^2 \Leftrightarrow$$

$$\Leftrightarrow x\left(1 - \frac{9}{16}x\right) = 0$$

$$\Leftrightarrow \cancel{x=0} \quad \vee \quad x = \frac{16}{9}$$

$\notin [1,+\infty[$.

$\text{Fix } f = \left\{\frac{16}{9}\right\}$

$$\lim_{n\to+\infty} f^n(2024) = \frac{16}{9}.$$

(all sequences of the type $(f^n(x_0))_n$ converge to the fixed point).

2a) $f(x,y) = xy - x^2 \ln y \qquad (x,y) \in D$

$$\nabla f(x,y) = \left(y - 2x \ln y ,\ x - \frac{x^2}{y} \right).$$

$$\nabla f(x,y) = \bar{0} \Leftrightarrow \begin{cases} y - 2x \ln y = 0 \\ x - \dfrac{x^2}{y} = 0 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ x\left(1 - \dfrac{x}{y}\right) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} x - 2x \ln x = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} x(1 - 2\ln x) = 0 \\ x = y \end{cases}$$

impossible

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} \ln x = \dfrac{1}{2} \\ y = \sqrt{e} \end{cases}.$$

$\boxed{\text{Critical point } (\sqrt{e}, \sqrt{e})}$

$$H_f(x,y) = \begin{pmatrix} -2\ln y & 1 - \dfrac{2x}{y} \\ 1 - \dfrac{2x}{y} & + \dfrac{x^2}{y^2} \end{pmatrix}$$

$$H_f(\sqrt{e}, \sqrt{e}) = \begin{pmatrix} -2 \cdot \frac{1}{2} & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$\Delta_1 = -1 < 0$ Q.F. UND $\Longrightarrow$ $(\sqrt{e}, \sqrt{e})$ is a saddle

$\Delta_2 = -1 - 1 = -2 < 0$ $\therefore$ There are no local extrema.

b)

$$\mathcal{L}(x,y,\lambda) = \underbrace{xy - x^2 \ln y}_{f(x,y)} - \lambda \underbrace{(xy-1)}_{g(x,y)}$$ ← restriction

$$\nabla \mathcal{L}(x,y,\lambda) = \vec{0} \Rightarrow \begin{cases} y - 2x\ln y - \lambda y = 0 \\ x - x^2 \cdot \frac{1}{y} - \lambda x = 0 \\ xy = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \underline{\qquad} \\ x\left(1 - \frac{x}{y} - \lambda\right) = 0 \\ xy = 1 \end{cases} \Rightarrow \begin{cases} y - 2x\ln y - \left(1 - \frac{x}{y}\right)y = 0 \\ \lambda = 1 - \frac{x}{y} \\ xy = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \cancel{y} - 2x\ln y - \cancel{y} + x = 0 \\ \underline{\qquad} \\ \underline{\qquad} \end{cases} \Leftrightarrow \begin{cases} x(1 - 2\ln y) = 0 \\ \underline{\qquad} \\ \underline{\qquad} \end{cases}$$

$$\Leftrightarrow \begin{cases} \ln y = 1/2 \\ \underline{\qquad} \\ \underline{\qquad} \end{cases} \Leftrightarrow \begin{cases} y = \sqrt{e} \\ x = e^{-1/2} \\ \lambda = 1 - \frac{e^{-1/2}}{e^{1/2}} \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{\sqrt{e}} \\ y = \sqrt{e} \\ \lambda = 1 - \frac{1}{e} \end{cases}$$

$$\underbrace{f\left(\frac{1}{\sqrt{e}}, \sqrt{e}\right)}_{\text{minimum}} = 1 - \frac{1}{e} \cdot \ln\sqrt{e} = 1 - \frac{1}{e} \cdot \frac{1}{2} = 1 - \frac{1}{2e}$$

⚠
- $\left(\frac{1}{\sqrt{e}}, \sqrt{e}\right)$ is the minimizer
- $f\left(\frac{1}{\sqrt{e}}, \sqrt{e}\right)$ is the minimum

3. $$xy' + y = xe^x$$

$$\overset{x\neq 0}{\Leftrightarrow} \quad y' + \frac{1}{x}y = e^x$$

Using the formula given in the lectures, we have:

$$y(x) = \frac{\int e^{\int \frac{1}{x}dx} \cdot e^x dx + C}{e^{\int \frac{1}{x}dx}} \Leftrightarrow$$

$$y(x) = \frac{\int e^{\ln x} \cdot e^x dx + C}{e^{\ln x}} \Leftrightarrow$$

$$y(x) = \frac{\int xe^x dx + C}{x} \quad C \in \mathbb{R} \Leftrightarrow$$

$$\Leftrightarrow y(x) = \frac{e^x x - e^x + C}{x} \Leftrightarrow$$

$$\Leftrightarrow y = e^x - \frac{e^x}{x} + \frac{C}{x}.$$

$$\int \underbrace{x}_{f} \underbrace{e^x}_{g'} dx = \underbrace{e^x}_{g} \cdot \underbrace{x}_{f} - \int \underbrace{e^x}_{g} \cdot \underbrace{1}_{f'} dx = e^x x - e^x$$

$y(1) = 1 \Leftrightarrow \quad 1 = e^1 - \frac{e^1}{1} + \frac{c}{1} \Leftrightarrow c = 1$

$\therefore \quad y(x) = e^x - \frac{e^x}{x} + \frac{1}{x} \quad , \; x \in \mathbb{R}^+$

max. domain.

4. $\begin{cases} \dot{x} = x + y \\ \dot{y} = -9x + y \end{cases} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

eigenvalues of A:

$P(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ -9 & 1-\lambda \end{pmatrix} \Leftrightarrow P(\lambda) = (1-\lambda)^2 + 9$

$P(\lambda) = 0 \Leftrightarrow 1 - \lambda = \pm 3i \Rightarrow \boxed{1 \pm 3i = \lambda} \in \mathbb{C} \setminus \mathbb{R}$

Eigenspaces

$E_{1+3i} \quad \begin{pmatrix} 1-(1+3i) & 1 \\ -9 & 1-(1+3i) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -3i & 1 \\ -9 & -3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -3ix + y = 0 \\ -9x - 3iy = 0 \end{cases}$

$\Leftrightarrow \begin{cases} y = 3ix \\ \text{—} \end{cases} \qquad E_{1+3i} = \langle (1, 3i) \rangle$

Eigenvectors: $E_{1+3i} \setminus \{(0,0)\}$

Solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{(1+3i)t} \begin{pmatrix} 1 \\ 3i \end{pmatrix} = e^{t}\left(\cos 3t + i \sin 3t\right) \begin{pmatrix} 1 \\ 3i \end{pmatrix}$$

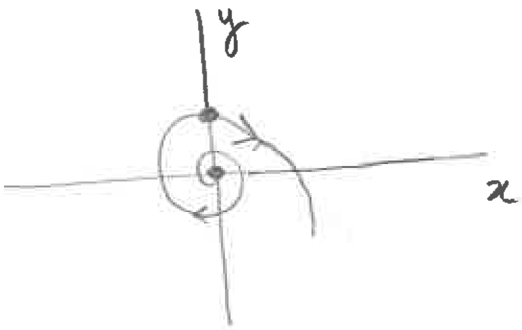
$$= \begin{pmatrix} e^{t} \left(\cos(3t) + i \sin(3t)\right) \\ e^{t} \left(3i\cos(3t) - 3 \sin(3t)\right) \end{pmatrix}$$

General solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 \, e^{t} \begin{pmatrix} \cos(3t) \\ -3\sin(3t) \end{pmatrix} + k_2 \, e^{t} \begin{pmatrix} \sin(3t) \\ 3\cos(3t) \end{pmatrix}$$

$k_1, k_2 \in \mathbb{R}$

b)



$(1,0) \longrightarrow$ tangent vector $(1,-9)$

5a) $F(t, x, \dot{x}) = 1 - x^2 - \dot{x}^2$

Euler-Lagrange equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \iff -2x - \frac{d}{dt}(-2\dot{x}) = 0$$

$$\iff -\cancel{2}x = -\cancel{2}\ddot{x}$$

E.L. equation

$$\boxed{\ddot{x} = x}$$

Transv. Condition

$$\frac{\partial F}{\partial \dot{x}}(1, x(1), \dot{x}(1)) = 0 \quad -2\dot{x}(1) = 0 \Leftrightarrow \dot{x}(1) = 0.$$

b) $\ddot{x} - x = 0$

$$P(\lambda) = \lambda^2 - 1 \quad ; \quad P(\lambda) = 0 \Leftrightarrow \lambda = \pm 1$$

$$x(t) = c_1 e^t + c_2 e^{-t} \qquad \dot{x}(t) = c_1 e^t - c_2 e^{-t}$$

$$\begin{cases} x(0) = 1 \\ \dot{x}(1) = 0 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 1 \\ c_1 e - c_2 e^{-1} = 0 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ c_1 e^2 = c_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 + c_1 e^2 = 1 \\ \text{———} \end{cases} \Leftrightarrow \begin{cases} c_1 = \dfrac{1}{1+e^2} \\ c_2 = \dfrac{e^2}{1+e^2} \end{cases}$$

$$\therefore x(t) = \frac{1}{1+e^2} e^t + \frac{e^2}{1+e^2} e^{-t}, \quad t \in [0,1)$$

Since F is concave in $(x, \dot{x})$, then $x(t)$ is in fact the solution.

$\downarrow$

$$F(t,x,\dot{x}) = 1 - x^2 - \dot{x}^2$$

$$\nabla F(x, \dot{x}) = (-2x, -2\dot{x})$$

$$HF(x,\dot{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

eigenvalues: $-2 < 0$

$\Downarrow$

H_f is N.D, $\forall (x,\dot{x}) \in A$

$\Downarrow$

f is concave